EFFECT OF THERMOMETER THERMAL CONDUCTIVITY AND DIMENSIONS ON ACCURACY OF THERMAL FLUX MEASUREMENTS

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Expressions are presented for evaluating errors in steady-state thermal flux measurements within massive objects with consideration of the effect of thermal conductivity and size of the active thermometer zone.

Gradient-type thermometers [1], constructed in the form of a disk or plate of limited size and installed within the object to be studied, distort the preexisting temperature and thermal flux distributions within the body. The amount of this distortion and, consequently, the error in the measurement of the steady-state thermal flux depend upon the ratio of the object's thermal conductivity to that of the thermometer and on the form and size of the thermometer.

An estimate of errors in thermometer indications produced by these factors was offered in [2], which used a solution of the problem of the dielectric electric field in the form of an ellipsoid of revolution located in the homogeneous electric field of another dielectric. Since objections have been raised regarding the adequacy of replacing real thermometer constructions (disks, plates of limited size) by an oblate ellipsoid of revolution, the present study will formulate thermometer thermal conductivity in a new manner in order to obtain calculation expressions for estimating errors in steady-state thermal flux measurements within massive objects with consideration of the actual dimensions and parameters of the thermometer active zone.

Let a gradient-type disk thermometer with radius R and thickness h, constructed of a material with thermal conductivity λ_T, be located within a massive object having a thermal conductivity λ. The initial temperature field in the object $t_0(z)$ is one-dimensional and characterized by the value of the temperature gradient b. Because of the difference between λ and λ_T in the zone where the thermometer is installed and the region adjacent thereto there are formed a new spatially inhomogeneous temperature distribution t(r, z) and thermal flux, symmetric about a plane which divides the thermometer into two halves each h/2 high (Fig. 1). The thermometer volume bounded by the radius R_e corresponds to the sensitive element.

Thermometers used in practice have a radius R several times greater than the thickness (usually R/h > 5). With consideration of this the following assumptions can be made: 1) the temperature perturbation in the upper part of the object (Fig. 1) is formed by transport through the thermal resistance produced by a layer of material h/2 thick which has a thermal conductivity λ_T within the radial coordinate limits $0 \le r \le R$ and λ at $R < r < \infty$; 2) heat transport in the radial direction is absent, i.e., the temperature field in the layer is linear in the coordinate z; 3) in light of the symmetry of the problem the lower plane of the layer (z = −h/2) has zero temperature, t(r, −h/2) = 0.

The initial (undistorted) temperature distribution within the object

$$t_0(z) = b\left(\frac{h}{2} + z\right). \tag{1}$$

The value of the temperature perturbation $\vartheta(r, z) = t(r, z) - t_0(z)$ is found by solution of the Laplace equation

$$\frac{\partial^2\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial\vartheta}{\partial r} + \frac{\partial^2\vartheta}{\partial z^2} = 0 \tag{2}$$

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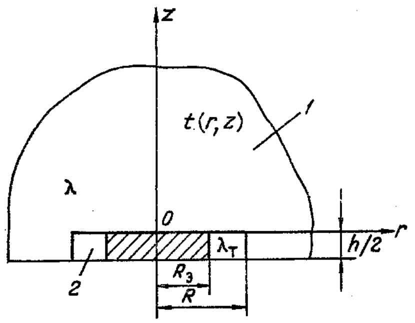

Fig. 1. Heat-exchange model: 1) object; 2) thermometer.

with boundary conditions

$$\vartheta(r,\ z)|_{z\to\infty}=0,\ \vartheta(r,\ z)|_{r\to\infty}=0, \tag{3}$$

$$\delta_\lambda\left[b+\frac{2}{h}\vartheta(r,\ 0)\right]\sigma(r-R)=\left(\frac{\partial\vartheta}{\partial z}\right)_{z=0}-\frac{2}{h}\vartheta(r,\ 0), \tag{4}$$

where $\delta_\lambda = (\lambda_T/\lambda - 1)$, and the symbol $\sigma(r - R)$ denotes the unit step function

$$\sigma(r-R)=\begin{cases}1, & 0\leqslant r<R,\\ 0, & r>R.\end{cases} \tag{5}$$

Taking the Hankel transform of Eq. (2) $L_H[\vartheta(r,\ z)] = \theta(p,\ z)$, with consideration of conditions (3) we find

$$\theta(p,\ z)=A(p)\exp(-pz). \tag{6}$$

To determine the value of A(p) one can use the transformed boundary condition Eq. (4):

$$\delta_\lambda\left[b+\frac{2}{h}\vartheta(r_*,\ 0)\right]\frac{RJ_1(pR)}{p}=\left(\frac{d\theta}{\partial z}\right)_{z=0}-\frac{2}{h}\theta(p,\ 0),\ 0\leqslant r_*\leqslant R, \tag{7}$$

in which the quantity $\vartheta(r_*,\ 0)$ which requires further definition is introduced using the mean value theorem for calculation of the integral

$$\int_0^\infty rJ_0(pr)\vartheta(r,\ 0)\sigma(r-R)\,dr=\vartheta(r_*,\ 0)\frac{RJ_1(pR)}{p}. \tag{8}$$

Substituting Eq. (6) in Eq. (7), with consideration of Eq. (8) we find

$$A(p)=-\delta_\lambda[bR+k\vartheta(r_*,\ 0)]\frac{J_1(pR)}{p\left(p+\frac{2}{h}\right)}, \tag{9}$$

where k = 2R/h.

Taking the reverse Hankel transform of Eq. (6) we obtain the desired equation for the value of the temperature perturbation

$$\vartheta(r,\ z)=-\delta_\lambda[bR+k\vartheta(r_*,\ 0)]\int_0^\infty\frac{J_1(pR)J_0(pr)\exp(-pz)\,dp}{p+\frac{2}{h}}. \tag{10}$$

In real constructions the thermometer sensitive element occupies only a portion of the volume bounded by the radius R_e. The annular portion of the thermometer ($R_e \le r \le R$) plays the role of a "guard zone," reducing the edge effect caused by thermometer presence within the body. Then the total thermal flux Q_e passing through the working portion of the thermometer with area $S = \pi R_e^2$ can be defined as

$$Q_e=\lambda_T\pi R_e^2\,\langle t(r,\ 0)\rangle\frac{2}{h}=\lambda_T\pi R_e^2\left\langle\left.\frac{dt(r,\ z)}{dz}\right|_{z=0}\right\rangle, \tag{11}$$

wherein we have the average over S of the temperature $\langle t(r,\ 0)\rangle$ or the temperature gradient $\left\langle\left.\frac{dt(r,\ z)}{dz}\right|_{z=0}\right\rangle$.

TABLE 1. Values of Integrals $\Phi(k, \rho_e)$ and $\Phi(k, 1)$ vs k and ρ_e

k	$\Phi(k,\rho_e)$										$\Phi(k,1)$
	ρ_e										
	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
4	0,221	0,222	0,223	0,227	0,231	0,237	0,246	0,257	0,273	0,295	0,332
6	0,156	0,157	0,158	0,161	0,164	0,170	0,177	0,187	0,201	0,223	0,262
10	0,097	0,098	0,099	0,101	0,103	0,107	0,112	0,120	0,131	0,150	0,189
20	0,050	0,050	0,050	0,051	0,053	0,055	0,058	0,062	0,069	0,082	0,116
40	0,025	0,025	0,025	0,026	0,027	0,028	0,029	0,032	0,035	0,043	0,069

As follows from Eq. (1) the undistorted (in the absence of the thermometer) value of the thermal flux Q_0 through the same area is:

$$Q_0 = \lambda\pi R_e^2 \frac{bh}{2}\,\frac{2}{h} = \lambda\pi R_e^2\, b. \tag{12}$$

The relative error of the thermal flux measurement can then be defined as

$$\delta(\rho_e) = \frac{Q_e - Q_0}{Q_0} = \delta_\lambda \left[1 + \frac{2}{bh}\,\frac{\delta_\lambda + 1}{\delta_\lambda}\langle \vartheta(\rho_e)\rangle\right]. \tag{13}$$

The mean value of the temperature distortion $\langle\vartheta(\rho_e)\rangle$ in the interval $0 \le r \le R_e$ can be found by solving Eq. (10) in which the value of the quantity $\vartheta(r_*, 0)$ is determined by the condition of satisfaction of boundary condition (4) on the average over the area πR_e^2. Finally, we have

$$\langle \vartheta(\rho_e)\rangle = -bR\delta_\lambda \int_0^\infty \frac{J_1(x)\,J_1(\rho_e x)\,dx}{x(x+k)} \Bigg/ \int_0^\infty \frac{[x + k(1+\delta_\lambda)]\,J_1(x)\,J_1(\rho_e x)\,dx}{x(x+k)}, \tag{14}$$

where $\rho_e = R_e/R$, $0 \le \rho_e \le 1$.

Substituting Eq. (14) in Eq. (13), we obtain an expression for estimating the error

$$\delta(\rho_e) = \delta_\lambda \frac{\Phi(k, \rho_e)}{1 + \delta_\lambda[1 - \Phi(k, \rho_e)]}, \tag{15}$$

where

$$\Phi(k, \rho_e) = \int_0^\infty \frac{J_1(x)\,J_1(\rho_e x)\,dx}{x+k} \Bigg/ \int_0^\infty \frac{J_1(x)\,J_1(\rho_e x)\,dx}{x}. \tag{16}$$

If the thermometer sensitive element occupies the entire surface, i.e., $\rho_e = R_e/R = 1$, then $\delta(\rho_e) = \delta(1)$ and

$$\Phi(k, 1) = 2\int_0^\infty \frac{J_1^2(x)\,dx}{x+k}. \tag{17}$$

The numerical values of the integrals of Eqs. (16), (17) as functions of k and ρ_e were calculated on a computer (Table 1) and then used to calculate relative error of thermometer readings with Eq. (15).

Results of calculating the error $\delta(1)$ for the case where the thermometer sensitive element occupies the entire surface ($\rho_e = 1$) are shown in Fig. 2. It is evident that the error increases severely as λ_T/λ departs from unity and decreases with increase in k. The effect of the size of the thermometer sensitive element ρ_e, calculated with the expression

$$\Delta = \frac{\delta(\rho_e)}{\delta(1)} = \frac{\Phi(k, \rho_e)}{\Phi(k, 1)}\,\frac{1 + \delta_\lambda[1 - \Phi(k, 1)]}{1 + \delta_\lambda[1 - \Phi(k, \rho_e)]}, \tag{18}$$

for the value k = 10 is shown in Fig. 3. It is evident that decrease in the size of the sensitive element from $\rho_e = 1$ to $\rho_e = 0.5$ leads to a reduction in the error $\delta(0.5)$ as compared to $\delta(1)$ by 30-48% for values of the ratio $\lambda_T/\lambda = 0.2$-10. In the limiting case $\lambda_T/\lambda = 1$ in accordance with Eq. (15) independent of the sensitive element size ρ_e the error $\delta(\rho_e) = 0$, since $\delta_\lambda = 0$.

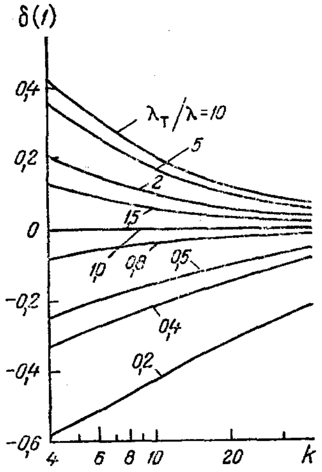

Fig. 2

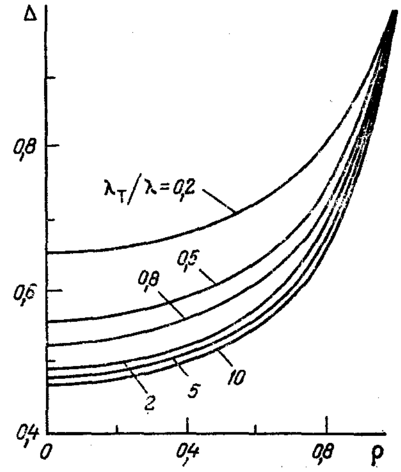

Fig. 3

Fig. 2. Error $\delta(1)$ of thermometer indication vs k for various values of ratio λ_T/λ.

Fig. 3. Effect of the size ρ_e of thermometer sensitive element on error for various values of λ_T/λ at k = 10.

TABLE 2. Ratio λ_T/λ vs Geometric Factor k for a Number of Error Levels $\delta(1)$

k	by Eq. (19), $\delta(1)$ ± 0,02	± 0,05	± 0,1	± 0,2
4	0,94...1,06	0,86...1,17	0,75...1,38	0,57...2,01
6	0,93...1,08	0,83...1,22	0,70...1,53	0,51...2,75
10	0,90...1,12	0,78...1,34	0,63...1,93	0,43...8,58
20	0,85...1,20	0,69...1,70	0,51...4,64	0,32...∞
40	0,77...1,40	0,57...3,23	0,38...∞	0,22...∞

k	by data of [2], $\delta(1)$ ± 0,02	± 0,05	± 0,1	± 0,2
4	0,94...1,07	0,85...1,19	0,73...1,44	0,54...2,29
6	0,91...1,10	0,80...1,28	0,66...1,73	0,46...4,42
10	0,87...1,16	0,73...1,52	0,56...2,88	0,36...∞
20	0,78...1,36	0,58...2,82	0,40...∞	0,23...∞
40	0,65...2,06	0,42...∞	0,26...∞	0,13...∞

For direct determination of the ratio λ_T/λ from Eq. (15) we have the expression

$$\frac{\lambda_T}{\lambda} = \left\{ 1 - \frac{\delta(\rho_e)}{\Phi(k,\ \rho_e)[1+\delta(\rho_e)]} \right\}^{-1}, \qquad (19)$$

which for specified values of measurement error $\delta(\rho_e)$ and thermometer geometric factor k allows us to find the permissible divergence in the thermal conductivities of the object λ and thermometer λ_T. Results of calculating $\delta(1)$ with Eq. (19) and data determined on the basis of [2] are presented in Table 2. The allowable divergence of λ_T from λ increases with increase in k and the allowable measurement uncertainty $\delta(1)$. Comparison of the data in Table 2 shows quite close agreement of the lower limits of the λ_T/λ range and appreciable differences in the upper λ_T/λ limits for large k. The symbol ∞ denotes limiting values of the λ_T/λ range not calculated with Eq. (19) or [2] in view of the approximateness of their derivation.

Analysis of the results obtained shows that providing an accuracy at the level $\delta(1) \leq 0.05$ for thermometer constructions used in practice with values k = 5-20 leads to quite severe restrictions on the thermal conductivity of the material used.

The size of the zone in which the temperature is perturbed by the presence of the thermometer can be estimated with Eq. (10). The expressions

$$\frac{\vartheta(r,\ 0)}{\vartheta(0,\ 0)} = \int\limits_0^\infty \frac{J_1(x)\, J_0\left(\frac{r}{R}\, x\right) dx}{x+k} \Bigg/ \int\limits_0^\infty \frac{J_1(x)\, dx}{x+k}, \tag{20}$$

$$\frac{\vartheta(0,\ z)}{\vartheta(0,\ 0)} = \int\limits_0^\infty \frac{J_1(x) \exp\left(-\frac{z}{R}\, x\right) dx}{x+k} \Bigg/ \int\limits_0^\infty \frac{J_1(x)\, dx}{x+k}$$

define the degree of damping of the perturbation in the directions r and z as compared to the maximum value $\vartheta(0, 0)$. Results of computer analysis reveal that for k = 20 the value of the ratio $\vartheta(0, z)/\vartheta(0, 0)$ for z/R = 1, 2, 10 is 0.33, 0.15, 0.015, respectively. The size of the perturbed zone is significantly less in the radial direction: the value of $\vartheta(r, 0)/\vartheta(0, 0)$ for k = 20 and r/R = 1, 2, 3 is 0.49, 0.004, 0.001.

Use of the relationships presented above permits a rational selection of thermometers for measurement of thermal flux in various media with consideration of the accuracy required.

NOTATION

r, z, axial and radial coordinates; r_*, r value for which mean value theorem is satisfied; λ, λ_T, thermal conductivities of object and thermometer, W/(m·K); R, h, thermometer radius and thickness, m; R_e, radius of thermometer sensitive element, m; t, temperature, K; b, temperature gradient, K/m; ϑ, temperature distortion, K; Q, thermal flux, W; δ, relative error in thermal flux measurement; p, Hankel transform parameter, m^{-1}; J_0, J_1, zeroth and first-order Bessel functions of the first sort.

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